# The Compositions of the Differential Operations and Gateaux Directional Derivative

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**Abstract**

In this paper we determine the number of the meaningful compositions of higher order of the differential operations and Gateaux directional derivative.

## 1 The compositions of the differential operations of the space $\mathbb{R}^3$

In the real three-dimensional space $\mathbb{R}^3$ we consider the following sets:

$$\mathrm{A}_0 = \{f:\mathbb{R}^3 \longrightarrow \mathbb{R} \mid f \in C^{\infty}(\mathbb{R}^3)\} \quad \text{and} \quad \mathrm{A}_1 = \{\vec{f}:\mathbb{R}^3 \longrightarrow \mathbb{R}^3 \mid \vec{f} \in \vec{C}^{\infty}(\mathbb{R}^3)\}. \tag{1}$$

Then, over the sets $\mathrm{A}_0$ and $\mathrm{A}_1$ in the vector analysis, there are $m = 3$ differential operations of the first-order:

$$\begin{aligned}
\operatorname{grad} f &= \nabla_1 f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right) : \mathrm{A}_0 \longrightarrow \mathrm{A}_1 \ , \\
\operatorname{curl} \vec{f} &= \nabla_2 \vec{f} = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) : \mathrm{A}_1 \longrightarrow \mathrm{A}_1 \ , \\
\operatorname{div} \vec{f} &= \nabla_3 \vec{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} : \mathrm{A}_1 \longrightarrow \mathrm{A}_0 \ .
\end{aligned} \tag{2}$$

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Let us present the number of the meaningful compositions of higher order over the set $\mathcal{A}_3 = \{\nabla_1, \nabla_2, \nabla_3\}$. As a well-known fact, there are $m = 5$ compositions of the second-order:

$$\begin{aligned}
&\Delta f = \operatorname{div}\operatorname{grad} f = \nabla_3 \circ \nabla_1 f,\\
&\operatorname{curl}\operatorname{curl} \vec{f} = \nabla_2 \circ \nabla_2 \vec{f},\\
&\operatorname{grad}\operatorname{div} \vec{f} = \nabla_1 \circ \nabla_3 \vec{f},\\
&\operatorname{curl}\operatorname{grad} f = \nabla_2 \circ \nabla_1 f = \vec{0},\\
&\operatorname{div}\operatorname{curl} \vec{f} = \nabla_3 \circ \nabla_2 \vec{f} = 0.
\end{aligned} \tag{3}$$

Malešević [2] proved that there are $m = 8$ compositions of the third-order:

$$\begin{aligned}
&\operatorname{grad}\operatorname{div}\operatorname{grad} f = \nabla_1 \circ \nabla_3 \circ \nabla_1 f,\\
&\operatorname{curl}\operatorname{curl}\operatorname{curl} \vec{f} = \nabla_2 \circ \nabla_2 \circ \nabla_2 \vec{f},\\
&\operatorname{div}\operatorname{grad}\operatorname{div} \vec{f} = \nabla_3 \circ \nabla_1 \circ \nabla_3 \vec{f},\\
&\operatorname{curl}\operatorname{curl}\operatorname{grad} f = \nabla_2 \circ \nabla_2 \circ \nabla_1 f = \vec{0},\\
&\operatorname{div}\operatorname{curl}\operatorname{grad} f = \nabla_3 \circ \nabla_2 \circ \nabla_1 f = 0,\\
&\operatorname{div}\operatorname{curl}\operatorname{curl} \vec{f} = \nabla_3 \circ \nabla_2 \circ \nabla_2 \vec{f} = 0,\\
&\operatorname{grad}\operatorname{div}\operatorname{curl} \vec{f} = \nabla_1 \circ \nabla_3 \circ \nabla_2 \vec{f} = \vec{0},\\
&\operatorname{curl}\operatorname{grad}\operatorname{div} \vec{f} = \nabla_2 \circ \nabla_1 \circ \nabla_3 \vec{f} = \vec{0}.
\end{aligned} \tag{4}$$

If we denote by $\mathtt{f}(k)$ the number of compositions of the $k^{\text{th}}$-order, then Malešević [3] proved:

$$\mathtt{f}(k) = F_{k+3}, \tag{5}$$

where $F_k$ is $k^{\text{th}}$ Fibonacci number.

# 2 The compositions of the differential operations and Gateaux directional derivative on the space $\mathbb{R}^3$

Let $f \in \mathrm{A}_0$ be a scalar function and $\vec{e} = (e_1, e_2, e_3) \in \mathbb{R}^3$ be a unit vector. Thus, the *Gateaux directional derivative* in direction $\vec{e}$ is defined by [1, p. 71]:

$$\operatorname{dir}_{\vec{e}} f = \nabla_0 f = \nabla_1 f \cdot \vec{e} = \frac{\partial f}{\partial x_1} e_1 + \frac{\partial f}{\partial x_2} e_2 + \frac{\partial f}{\partial x_3} e_3 : \mathrm{A}_0 \longrightarrow \mathrm{A}_0. \tag{6}$$

Let us determine the number of the meaningful compositions of higher order over the set $\mathcal{B}_3 = \{\nabla_0, \nabla_1, \nabla_2, \nabla_3\}$. There exist $m = 8$ compositions of the second-order:

$$
\begin{aligned}
&\operatorname{dir}_{\vec{e}} \operatorname{dir}_{\vec{e}} f = \nabla_0 \circ \nabla_0 f = \nabla_1\big(\nabla_1 f \cdot \vec{e}\big) \cdot \vec{e},\\
&\operatorname{grad} \operatorname{dir}_{\vec{e}} f = \nabla_1 \circ \nabla_0 f = \nabla_1\big(\nabla_1 f \cdot \vec{e}\big),\\
&\Delta f = \operatorname{div} \operatorname{grad} f = \nabla_3 \circ \nabla_1 f,\\
&\operatorname{curl} \operatorname{curl} \vec{f} = \nabla_2 \circ \nabla_2 \vec{f},\\
&\operatorname{dir}_{\vec{e}} \operatorname{div} \vec{f} = \nabla_0 \circ \nabla_3 \vec{f} = \big(\nabla_1 \circ \nabla_3 \vec{f}\big) \cdot \vec{e},\\
&\operatorname{grad} \operatorname{div} \vec{f} = \nabla_1 \circ \nabla_3 \vec{f},\\
&\operatorname{curl} \operatorname{grad} f = \nabla_2 \circ \nabla_1 f = \vec{0},\\
&\operatorname{div} \operatorname{curl} \vec{f} = \nabla_3 \circ \nabla_2 \vec{f} = 0;
\end{aligned}
\tag{7}
$$

that is, there exist $m = 16$ compositions of the third-order:

$$
\begin{aligned}
&\operatorname{dir}_{\vec{e}} \operatorname{dir}_{\vec{e}} \operatorname{dir}_{\vec{e}} f = \nabla_0 \circ \nabla_0 \circ \nabla_0 f,\\
&\operatorname{grad} \operatorname{dir}_{\vec{e}} \operatorname{dir}_{\vec{e}} f = \nabla_1 \circ \nabla_0 \circ \nabla_0 f,\\
&\operatorname{div} \operatorname{grad} \operatorname{dir}_{\vec{e}} f = \nabla_3 \circ \nabla_1 \circ \nabla_0 f,\\
&\operatorname{dir}_{\vec{e}} \operatorname{div} \operatorname{grad} f = \nabla_0 \circ \nabla_3 \circ \nabla_1 f,\\
&\operatorname{grad} \operatorname{div} \operatorname{grad} f = \nabla_1 \circ \nabla_3 \circ \nabla_1 f,\\
&\operatorname{curl} \operatorname{curl} \operatorname{curl} \vec{f} = \nabla_2 \circ \nabla_2 \circ \nabla_2 \vec{f},\\
&\operatorname{dir}_{\vec{e}} \operatorname{dir}_{\vec{e}} \operatorname{div} \vec{f} = \nabla_0 \circ \nabla_0 \circ \nabla_3 \vec{f},\\
&\operatorname{grad} \operatorname{dir}_{\vec{e}} \operatorname{div} \vec{f} = \nabla_1 \circ \nabla_0 \circ \nabla_3 \vec{f},\\
&\operatorname{div} \operatorname{grad} \operatorname{div} \vec{f} = \nabla_3 \circ \nabla_1 \circ \nabla_3 \vec{f},\\
&\operatorname{curl} \operatorname{grad} \operatorname{dir}_{\vec{e}} f = \nabla_2 \circ \nabla_1 \circ \nabla_0 \vec{f} = \vec{0},\\
&\operatorname{curl} \operatorname{curl} \operatorname{grad} f = \nabla_2 \circ \nabla_2 \circ \nabla_1 f = \vec{0},\\
&\operatorname{div} \operatorname{curl} \operatorname{grad} f = \nabla_3 \circ \nabla_2 \circ \nabla_1 f = 0,\\
&\operatorname{div} \operatorname{curl} \operatorname{curl} \vec{f} = \nabla_3 \circ \nabla_2 \circ \nabla_2 \vec{f} = 0,\\
&\operatorname{dir}_{\vec{e}} \operatorname{div} \operatorname{curl} \vec{f} = \nabla_0 \circ \nabla_3 \circ \nabla_2 \vec{f} = 0,\\
&\operatorname{grad} \operatorname{div} \operatorname{curl} \vec{f} = \nabla_1 \circ \nabla_3 \circ \nabla_2 \vec{f} = \vec{0},\\
&\operatorname{curl} \operatorname{grad} \operatorname{div} \vec{f} = \nabla_2 \circ \nabla_1 \circ \nabla_3 \vec{f} = \vec{0}.
\end{aligned}
\tag{8}
$$

Using the method from the paper [3] let us define a binary relation $\sigma$ "*to be in composition*": $\nabla_i \, \sigma \, \nabla_j = \top$ iff the composition $\nabla_j \circ \nabla_i$ is meaningful. Thus, Cayley table of the relation $\sigma$ is determined with

$$\begin{array}{c|cccc} \sigma & \nabla_0 & \nabla_1 & \nabla_2 & \nabla_3 \\ \hline \nabla_0 & \top & \top & \bot & \bot \\ \nabla_1 & \bot & \bot & \top & \top \\ \nabla_2 & \bot & \bot & \top & \top \\ \nabla_3 & \top & \top & \bot & \bot \end{array} \tag{9}$$

Let us form the graph according to the following rule: if $\nabla_i \, \sigma \, \nabla_j = \top$ let vertex $\nabla_j$ be under vertex $\nabla_i$ and let there exist an edge from the vertex $\nabla_i$ to the vertex $\nabla_j$. Further on, let us denote by $\nabla_{-1}$ nowhere-defined function $\vartheta$, where domain and range are the empty sets [2]. We shall define $\nabla_{-1} \, \sigma \, \nabla_i = \top$ $(i = 0, 1, 2, 3, 4)$. For the set $\mathcal{B}_3 \cup \{\nabla_{-1}\}$ the graph of the walks, determined previously, is a tree with the root in the vertex $\nabla_{-1}$.

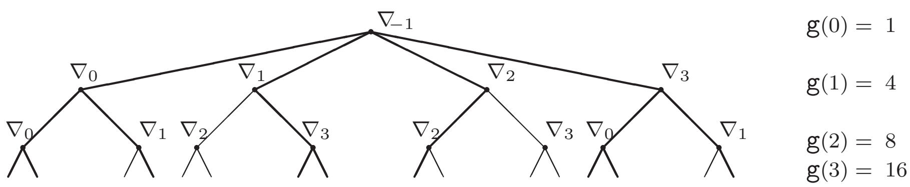


Fig. 1

Let $\mathsf{g}(k)$ be the number of the meaningful compositions of the $k^{\text{th}}$-order of the functions from $\mathcal{B}_3$. Let $\mathsf{g}_i(k)$ be the number of the meaningful compositions of the $k^{\text{th}}$-order beginning from the left by $\nabla_i$. Then $\mathsf{g}(k) = \mathsf{g}_0(k) + \mathsf{g}_1(k) + \mathsf{g}_2(k) + \mathsf{g}_3(k)$. Based on the partial self similarity of the tree (Fig. 1) we get equalities

$$\begin{aligned} \mathsf{g}_0(k) &= \mathsf{g}_0(k-1) + \mathsf{g}_1(k-1), \\ \mathsf{g}_1(k) &= \mathsf{g}_2(k-1) + \mathsf{g}_3(k-1), \\ \mathsf{g}_2(k) &= \mathsf{g}_2(k-1) + \mathsf{g}_3(k-1), \\ \mathsf{g}_3(k) &= \mathsf{g}_0(k-1) + \mathsf{g}_1(k-1). \end{aligned} \tag{10}$$

Hence, a recurrence for $\mathsf{g}(k)$ can be derived as follows:

$$\mathsf{g}(k) = 2\,\mathsf{g}(k-1). \tag{11}$$

Based on the initial value $\mathsf{g}(1) = 4$, we can conclude:

$$\mathsf{g}(k) = 2^{k+1}. \tag{12}$$

## 3 The compositions of the differential operations of the space $\mathbb{R}^{\mathbf{n}}$

Let us present the number of the meaningful compositions of differential operations in the vector analysis of the space $\mathbb{R}^n$, where differential operations $\nabla_r$ $(r=1,\ldots,n)$ are defined over non-empty corresponding sets $\mathrm{A}_s$ $(s=1,\ldots,m$ and $m=\lfloor n/2\rfloor$, $n\geq 3)$ according to the papers [3], [4]:

$$
\begin{array}{ll}
\mathcal{A}_n\ (n=2m)\!: & \nabla_1 : \mathrm{A}_0 \to \mathrm{A}_1 \\
 & \nabla_2 : \mathrm{A}_1 \to \mathrm{A}_2 \\
 & \vdots \\
 & \nabla_i : \mathrm{A}_{i-1} \to \mathrm{A}_i \\
 & \vdots \\
 & \nabla_m : \mathrm{A}_{m-1} \to \mathrm{A}_m \\
 & \nabla_{m+1} : \mathrm{A}_m \to \mathrm{A}_{m-1} \\
 & \vdots \\
 & \nabla_{n-j} : \mathrm{A}_{j+1} \to \mathrm{A}_j \\
 & \vdots \\
 & \nabla_{n-1} : \mathrm{A}_2 \to \mathrm{A}_1 \\
 & \nabla_n : \mathrm{A}_1 \to \mathrm{A}_0,
\end{array}
\qquad
\begin{array}{ll}
\mathcal{A}_n\ (n=2m+1)\!: & \nabla_1 : \mathrm{A}_0 \to \mathrm{A}_1 \\
 & \nabla_2 : \mathrm{A}_1 \to \mathrm{A}_2 \\
 & \vdots \\
 & \nabla_i : \mathrm{A}_{i-1} \to \mathrm{A}_i \\
 & \vdots \\
 & \nabla_m : \mathrm{A}_{m-1} \to \mathrm{A}_m \\
 & \nabla_{m+1} : \mathrm{A}_m \to \mathrm{A}_m \\
 & \nabla_{m+2} : \mathrm{A}_m \to \mathrm{A}_{m-1} \\
 & \vdots \\
 & \nabla_{n-j} : \mathrm{A}_{j+1} \to \mathrm{A}_j \\
 & \vdots \\
 & \nabla_{n-1} : \mathrm{A}_2 \to \mathrm{A}_1 \\
 & \nabla_n : \mathrm{A}_1 \to \mathrm{A}_0.
\end{array}
\tag{13}
$$

Let us define *higher order differential operations* as the meaningful compositions of higher order of differential operations from the set $\mathcal{A}_n = \{\nabla_1,\ldots,\nabla_n\}$. The number of the higher order differential operations is given according to the paper [3]. Let us define a binary relation $\rho$ "*to be in composition*": $\nabla_i\,\rho\,\nabla_j = \top$ iff the composition $\nabla_j \circ \nabla_i$ is meaningful. Thus, Cayley table of the relation $\rho$ is determined with

$$
\nabla_i\,\rho\,\nabla_j = \begin{cases} \top \ , & (j=i+1) \vee (i+j=n+1); \\ \bot \ , & \text{otherwise.} \end{cases} \tag{14}
$$

Let us form the adjacency matrix $\mathtt{A} = [a_{ij}] \in \{0,1\}^{n\times n}$ associated with the graph, which is determined by the relation $\rho$. Thus, according to the paper [4], the following statement is true.

**Theorem 3.1.** *Let $P_n(\lambda)=|\mathtt{A}-\lambda\mathtt{I}|=\alpha_0\lambda^n+\alpha_1\lambda^{n-1}+\cdots+\alpha_n$ be the characteristic polynomial of the matrix $\mathtt{A}$ and $v_n=[\,1\ \ldots\ 1\,]_{1\times n}$. If we denote by $\mathtt{f}(k)$ the number of the $k^{th}$-order differential operations, then the following formulas are true*:

$$
\mathtt{f}(k) = v_n \cdot \mathtt{A}^{k-1} \cdot v_n^T \tag{15}
$$

*and*

$$
\alpha_0\mathtt{f}(k) + \alpha_1\mathtt{f}(k-1) + \cdots + \alpha_n\mathtt{f}(k-n) = 0 \quad (k>n). \tag{16}
$$

**Lemma 3.2.** *Let $P_n(\lambda)$ be the characteristic polynomial of the matrix* $\mathtt{A}$. *Then the following recurrence is true*:

$$P_n(\lambda) = \lambda^2\big(P_{n-2}(\lambda) - P_{n-4}(\lambda)\big). \tag{17}$$

**Lemma 3.3.** *Let $P_n(\lambda)$ be the characteristic polynomial of the matrix* $\mathtt{A}$. *Then it has the following explicit representation:*

$$P_n(\lambda) = \begin{cases} \displaystyle\sum_{k=1}^{\lfloor \frac{n+2}{4} \rfloor + 1} (-1)^{k-1} \binom{\frac{n}{2}-k+2}{k-1} \lambda^{n-2k+2} & , \ n=2m; \\ \displaystyle\sum_{k=1}^{\lfloor \frac{n+2}{4} \rfloor + 2} (-1)^{k-1} \left( \binom{\frac{n+3}{2}-k}{k-1} + \binom{\frac{n+3}{2}-k}{k-2} \lambda \right) \lambda^{n-2k+2} & , \ n=2m+1. \end{cases} \tag{18}$$

The number of the higher order differential operations is determined by corresponding recurrence, which for dimension $n = 3, 4, 5, \ldots, 10$ , we refer according to [3]:

| Dimension: | Recurrence for the number of the $k^{\text{th}}$-order differential operations: |
|---|---|
| $n = 3$ | $\mathtt{f}(k) = \mathtt{f}(k-1) + \mathtt{f}(k-2)$ |
| $n = 4$ | $\mathtt{f}(k) = 2\mathtt{f}(k-2)$ |
| $n = 5$ | $\mathtt{f}(k) = \mathtt{f}(k-1) + 2\mathtt{f}(k-2) - \mathtt{f}(k-3)$ |
| $n = 6$ | $\mathtt{f}(k) = 3\mathtt{f}(k-2) - \mathtt{f}(k-4)$ |
| $n = 7$ | $\mathtt{f}(k) = \mathtt{f}(k-1) + 3\mathtt{f}(k-2) - 2\mathtt{f}(k-3) - \mathtt{f}(k-4)$ |
| $n = 8$ | $\mathtt{f}(k) = 4\mathtt{f}(k-2) - 3\mathtt{f}(k-4)$ |
| $n = 9$ | $\mathtt{f}(k) = \mathtt{f}(k-1) + 4\mathtt{f}(k-2) - 3\mathtt{f}(k-3) - 3\mathtt{f}(k-4) + \mathtt{f}(k-5)$ |
| $n = 10$ | $\mathtt{f}(k) = 5\mathtt{f}(k-2) - 6\mathtt{f}(k-4) + \mathtt{f}(k-6)$ |

For considered dimensions $n = 3, 4, 5, \ldots, 10$, the values of the function $\mathtt{f}(k)$, for small values of the argument $k$, are given in the database of integer sequences [6] as sequences A020701 ($n = 3$), A090989 ($n = 4$), A090990 ($n = 5$), A090991 ($n = 6$), A090992 ($n = 7$), A090993 ($n = 8$), A090994 ($n = 9$), A090995 ($n = 10$), respectively.

# 4 The compositions of the differential operations and Gateaux directional derivative of the space $\mathbb{R}^{\mathbf{n}}$

Let $f \in A_0$ be a scalar function and $\vec{e} = (e_1, \ldots, e_n) \in \mathbb{R}^n$ be a unit vector. Thus, the *Gateaux directional derivative* in direction $\vec{e}$ is defined by [1, p. 71]:

$$\operatorname{dir}_{\vec{e}} f = \nabla_0 f = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} e_k : A_0 \longrightarrow A_0. \tag{19}$$

Let us extend the set of differential operations $\mathcal{A}_n = \{\nabla_1, \ldots, \nabla_n\}$ with Gateaux directional derivational to the set $\mathcal{B}_n = \mathcal{A}_n \cup \{\nabla_0\} = \{\nabla_0, \nabla_1, \ldots, \nabla_n\}$:

$$\mathcal{B}_n\ (n{=}2m){:}\ \begin{array}{l} \nabla_0 : \mathrm{A}_0 \to \mathrm{A}_0 \\ \nabla_1 : \mathrm{A}_0 \to \mathrm{A}_1 \\ \nabla_2 : \mathrm{A}_1 \to \mathrm{A}_2 \\ \vdots \\ \nabla_i : \mathrm{A}_{i-1} \to \mathrm{A}_i \\ \vdots \\ \nabla_m : \mathrm{A}_{m-1} \to \mathrm{A}_m \\ \nabla_{m+1} : \mathrm{A}_m \to \mathrm{A}_{m-1} \\ \vdots \\ \nabla_{n-j} : \mathrm{A}_{j+1} \to \mathrm{A}_j \\ \vdots \\ \nabla_{n-1} : \mathrm{A}_2 \to \mathrm{A}_1 \\ \nabla_n : \mathrm{A}_1 \to \mathrm{A}_0, \end{array} \qquad \mathcal{B}_n\ (n{=}2m{+}1){:}\ \begin{array}{l} \nabla_0 : \mathrm{A}_0 \to \mathrm{A}_0 \\ \nabla_1 : \mathrm{A}_0 \to \mathrm{A}_1 \\ \nabla_2 : \mathrm{A}_1 \to \mathrm{A}_2 \\ \vdots \\ \nabla_i : \mathrm{A}_{i-1} \to \mathrm{A}_i \\ \vdots \\ \nabla_m : \mathrm{A}_{m-1} \to \mathrm{A}_m \\ \nabla_{m+1} : \mathrm{A}_m \to \mathrm{A}_m \\ \nabla_{m+2} : \mathrm{A}_m \to \mathrm{A}_{m-1} \\ \vdots \\ \nabla_{n-j} : \mathrm{A}_{j+1} \to \mathrm{A}_j \\ \vdots \\ \nabla_{n-1} : \mathrm{A}_2 \to \mathrm{A}_1 \\ \nabla_n : \mathrm{A}_1 \to \mathrm{A}_0. \end{array} \tag{20}$$

Let us define *higher order differential operations with Gateaux derivative* as the meaningful compositions of higher order of the functions from the set $\mathcal{B}_n = \{\nabla_0, \nabla_1, \ldots, \nabla_n\}$. We determine the number of the higher order differential operations with Gateaux derivative by defining a binary relation $\sigma$ "*to be in composition*":

$$\nabla_i\, \sigma\, \nabla_j = \begin{cases} \top\ , & (i{=}0 \wedge j{=}0) \vee (i{=}n \wedge j{=}0) \vee (j{=}i{+}1) \vee (i{+}j{=}n{+}1); \\ \bot\ , & \text{otherwise.} \end{cases} \tag{21}$$

Let us form the adjacency matrix $\mathtt{B} = [b_{ij}] \in \{0, 1\}^{(n+1)\times n}$ associated with the graph, which is determined by relation $\sigma$. Thus, analogously to the paper [4], the following statement is true.

**Theorem 4.1.** *Let $Q_n(\lambda) = |\mathtt{B} - \lambda \mathtt{I}| = \beta_0 \lambda^{n+1} + \beta_1 \lambda^n + \cdots + \beta_{n+1}$ be the characteristic polynomial of the matrix $\mathtt{B}$ and $v_{n+1} = [\,1\ \ldots\ 1\,]_{1\times(n+1)}$. If we denote by $\mathsf{g}(k)$ the number of the $k^{th}$-order differential operations with Gateaux derivative, then the following formulas are true*:

$$\mathsf{g}(k) = v_{n+1} \cdot \mathtt{B}^{k-1} \cdot v_{n+1}^T \tag{22}$$

*and*

$$\beta_0 \mathsf{g}(k) + \beta_1 \mathsf{g}(k-1) + \cdots + \beta_{n+1} \mathsf{g}(k - (n+1)) = 0 \quad (k > n{+}1). \tag{23}$$

**Lemma 4.2.** *Let $Q_n(\lambda)$ and $P_n(\lambda)$ be the characteristic polynomials of the matrices $\mathtt{B}$ and $\mathtt{A}$ respectively. Then the following equality is true*:

$$Q_n(\lambda) = \lambda^2 P_{n-2}(\lambda) - \lambda P_n(\lambda). \tag{24}$$

**Proof.** Let us determine the characteristic polynomial $Q_n(\lambda) = |\mathtt{B} - \lambda \mathtt{I}|$ by

$$Q_n(\lambda) = \begin{vmatrix} 1-\lambda & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 0 & -\lambda & 1 & \dots & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & \dots & 0 & -\lambda & 1 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & -\lambda & 1 \\ 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & -\lambda \end{vmatrix}. \tag{25}$$

Expanding the determinant $Q_n(\lambda)$ by the first column we have

$$Q_n(\lambda) = (1-\lambda)P_n(\lambda) + (-1)^{n+2}D_n(\lambda), \tag{26}$$

where is

$$D_n(\lambda) = \begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -\lambda & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & -\lambda & 1 & 0 & \dots & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & \dots & -\lambda & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & -\lambda & 1 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & -\lambda & 1 \end{vmatrix}. \tag{27}$$

Let us expand the determinant $D_n(\lambda)$ by the first row and then, in the next step, let us multiply the first row by $-1$ and add it to the last row. Then, we obtain the determinant of order $n-1$ :

$$D_n(\lambda) = \begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ -\lambda & 1 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & -\lambda & 1 & 0 & \dots & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 & \dots & -\lambda & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & -\lambda & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -\lambda & 0 \end{vmatrix}. \tag{28}$$

Expanding the previous determinant by the last column we have

$$D_n(\lambda) = (-1)^n \begin{vmatrix} -\lambda & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & -\lambda & 1 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & -\lambda & 1 & \dots & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 & \dots & 0 & -\lambda & 1 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & -\lambda & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -\lambda \end{vmatrix}. \tag{29}$$

If we expand the previous determinant by the last row, and if we expand the obtained determinant by the first column, we have the determinant of order $n-4$ :

$$D_n(\lambda) = (-1)^n\lambda^2 \begin{vmatrix} -\lambda & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & -\lambda & 1 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & -\lambda & 1 & \dots & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 & \dots & 0 & -\lambda & 1 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -\lambda \end{vmatrix}. \tag{30}$$

In other words

$$D_n(\lambda) = (-1)^n\lambda^2 P_{n-4}(\lambda). \tag{31}$$

¿From equalities (31) and (26) there follows:

$$Q_n(\lambda) = (1-\lambda)P_n(\lambda) + \lambda^2 P_{n-4}(\lambda). \tag{32}$$

On the basis of Lemma 3.2. the following equality is true:

$$Q_n(\lambda) = \lambda^2 P_{n-2}(\lambda) - \lambda P_n(\lambda). \ \blacksquare \tag{33}$$

**Lemma 4.3.** *Let $Q_n(\lambda)$ be the characteristic polynomial of the matrix* $\mathsf{B}$. *Then the following recurrence is true*:

$$Q_n(\lambda) = \lambda^2\big(Q_{n-2}(\lambda) - Q_{n-4}(\lambda)\big). \tag{34}$$

**Proof.** On the basis of Lemma 3.2. and Lemma 4.2. there follows the statement. $\blacksquare$

**Lemma 4.4.** *Let $Q_n(\lambda)$ be the characteristic polynomial of the matrix* $\mathsf{B}$. *Then it has the following explicit representation*:

$$Q_n(\lambda) = \begin{cases} (\lambda-2)\displaystyle\sum_{k=1}^{\lfloor\frac{n}{4}\rfloor+1} (-1)^{k-1}\binom{\frac{n+1}{2}-k}{k-1}\lambda^{n-2k+2} & , \ n{=}2m{+}1; \\ \displaystyle\sum_{k=1}^{\lfloor\frac{n+3}{4}\rfloor+2} (-1)^{k-1}\left(\binom{\frac{n}{2}-k+2}{k-1} + \binom{\frac{n}{2}-k+2}{k-2}\lambda\right)\lambda^{n-2k+3} & , \ n{=}2m. \end{cases} \tag{35}$$

**Proof.** On the basis of Lemma 3.3 and Lemma 4.2. there follows the statement. $\blacksquare$

The number of the higher order differential operations with Gateaux derivative is determined by corresponding recurrences, which for dimension $n = 3, 4, 5, \ldots, 10$, we can get by the means of [5]:

| Dimension: | Recurrence for the num. of the $k^{\text{th}}$-order diff. operations with Gateaux derivative: |
|---|---|
| $n = 3$ | $\mathsf{g}(k) = 2\mathsf{g}(k-1)$ |
| $n = 4$ | $\mathsf{g}(k) = \mathsf{g}(k-1) + 2\mathsf{g}(k-2) - \mathsf{g}(k-3)$ |
| $n = 5$ | $\mathsf{g}(k) = 2\mathsf{g}(k-1) + \mathsf{g}(k-2) - 2\mathsf{g}(k-3)$ |
| $n = 6$ | $\mathsf{g}(k) = \mathsf{g}(k-1) + 3\mathsf{g}(k-2) - 2\mathsf{g}(k-3) - \mathsf{g}(k-4)$ |
| $n = 7$ | $\mathsf{g}(k) = 2\mathsf{g}(k-1) + 2\mathsf{g}(k-2) - 4\mathsf{g}(k-3)$ |
| $n = 8$ | $\mathsf{g}(k) = \mathsf{g}(k-1) + 4\mathsf{g}(k-2) - 3\mathsf{g}(k-3) - 3\mathsf{g}(k-4) + \mathsf{g}(k-5)$ |
| $n = 9$ | $\mathsf{g}(k) = 2\mathsf{g}(k-1) + 3\mathsf{g}(k-2) - 6\mathsf{g}(k-3) - \mathsf{g}(k-4) + 2\mathsf{g}(k-5)$ |
| $n = 10$ | $\mathsf{g}(k) = \mathsf{g}(k-1) + 5\mathsf{g}(k-2) - 4\mathsf{g}(k-3) - 6\mathsf{g}(k-4) + 3\mathsf{g}(k-5) + \mathsf{g}(k-6)$ |

For considered dimensions $n = 3, 4, 5, \ldots, 10$, the values of the function $\mathsf{g}(k)$, for small values of the argument $k$, are given in the database of integer sequences [6] as sequences A000079 ($n = 3$), A090990 ($n = 4$), A007283 ($n = 5$), A090992 ($n = 6$), A000079 ($n = 7$), A090994 ($n = 8$), A020714 ($n = 9$), A129638 ($n = 10$), respectively.

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